# INTERACTION BETWEEN A VORTEX AND A PLANE

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Gol'dshtik [1] discussed the problem of the interaction between a vortex and a plane in a viscous liquid and established that a finite solution of this problem exists only at small Reynolds numbers. This result has been analyzed, for example, in [2-7]. The present article is devoted to an evaluation of [7], which contains the most complete analytical results. In [7] the concept of the solution was considerably broadened, as a result of which the results of [1] were found to be a partial case of a broad class of solutions, depending on an arbitrary parameter and existing with the appropriate choice of this parameter, at any arbitrary Reynolds numbers.

The statement of the problem consists in the following: a solution is sought to Navier–Stokes equations of the form

$$v_R = \frac{F'(x)}{R}, \quad v_\alpha = \frac{F(x)}{r}, \quad v_\theta = \frac{\Omega(x)}{r} \tag{1}$$

where $v_R$, $v_\alpha$, $v_\theta$ are the components of the velocity vector in a spherical system of coordinates $(R, \alpha, \theta)$; $r = R \sin\alpha$; $x = \cos\alpha$; $\alpha$ is the azimuthal angle. (Here and in what follows, the notation of Serrin [7] is used.) After substitution of (1) into the Navier–Stokes equations is obtained a system of ordinary differential equations

$$\nu(1-x^2)F^{\mathrm{IV}} - 4\nu x F''' + FF''' + 3F'F'' = -2\Omega\Omega' / (1 = x^2) \tag{2}$$

$$\nu(1-x^2)\Omega'' + F\Omega' = 0 \tag{3}$$

where $\nu$ is the coefficient of kinematic viscosity.

For this system of the sixth order five boundary conditions are imposed

$$\Omega(0) = F(0) = F'(0) = 0, \quad \Omega(1) = C, \quad F(1) = 0$$

By additional transformations we obtain the system

$$f' + f^2 = k^2 G(x) / (1 - x^2)^2. \tag{4}$$

$$\Omega'' + 2f\Omega' = 0 \tag{5}$$

$$G(x) = 2(1-x)^2 \int_0^x \frac{t\Omega^2 dt}{(1-t^2)^2} + 2x \int_x^1 \frac{\Omega^2 dt}{(1+t)^2} - P(x - x^2) \tag{6}$$

$$\begin{gathered} k = {}^1/_2 \nu^{-1}, \quad F = 2\nu(1-x^2) f \\ f(0) = \Omega(0), \quad \Omega(1) = C \end{gathered} \tag{7}$$

Here P is a free parameter, whose origin is bound up with the insufficient number of boundary conditions for the system (2)-(3).

In [1] the statement of the problem contained no kind of arbitrariness. Here, the requirement for the finite nature of the longitudinal velocity $v_R$ at the vortical line $(x = 1)$ was used as an insufficient condition. This requirement generated the following conditions: $F'(1)$ is finite; $f(1)$ is finite; $G'(1) = 0$. The latter equality determined the parameter $P = C^2$.

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The introduction of the parameter P, broadening the class of solutions, requires clarification of its physical sense. Therefore, it is advisable to clarify the consequence of the introduction of the arbitrary parameter P, and to furnish an interpretation of the results obtained.

First of all, with an arbitrary value of P, the function $v_R$ becomes infinite, having a logarithmic discontinuity [7]

$$Rv_R \sim -{}^1/_2\,(1 - P^*)\,k^* \ln (1 - x)^{-1} \tag{8}$$

Here $P^*$ and $k^*$ are normalized values of P and k

$$P = C^2 P^*, \quad k^* = |\,C\,|\,k$$

It must be noted that a similar type of singular solutions for self-similar flows of type (1) is well known in the theory of viscous jets. For example, in [8] it is shown that the solution of L. D. Landau with respect to a submerged jet becomes nonsingular. However, the author of this article assumes that singular solutions do not have physical meanings. The same kind of assertion is made in [9, 10]. However, in the theory of jets finite solutions exist at any arbitrary values of the parameters, while, for the problem of a vortex, the condition of boundedness is found to be too severe; therefore, the use of unbounded solutions may turn out to be justified.

However, if we assume that the parameter P is completely arbitrary, we obtain the result that the motion of the liquid is conserved, even with the disappearance of the vortex, when $C \to 0$. Actually, this follows from [7], in which the following equation is discussed:

$$f' + f^2 = -Pk^2x\,/\,(1 - x)\,(1 + x)^2, \quad f(0) = 0$$

which follows from the system (4)-(6) with $\Omega \equiv 0$. It is established in [7] that this equation is solvable if $Pk^2 < 8.2$. We point out that in [7] this equation contains the factor $P^*k^{*2}$, made up of normalized parameters; however, it can be replaced by the factor $Pk^2$, since this product does not depend on C.

Thus, an arbitrary value of P corresponds not to a "pure vortex," but to its superposition on the flow of the liquid, which is induced by the longitudinal motion of an infinitely thin filament at an infinitely great velocity. In distinction from motion at a finite velocity, such a filament is found capable of entraining a viscous liquid. Under these circumstances, still another finite momentum is introduced into the liquid, so that there is obtained a still more complex superposition of vortex, "filament," and jet.

Since this is so, it is clear that the additional sources of the motion of the liquid demands the assignment of parameters determining their intensity; for example, in locating a linear source or sink at the axis of a vortex, it is necessary to characterize its assigned strength. The choice of a type of singularity must be determined by a real physical problem which is such that the solution with singularities under consideration is asymptotic. Specifically, an attempt may be made to interpret solution (1) to the problem of a vortex as asymptotic for the motion of a viscous liquid due to a rotating needle, when it diameter approaches zero, while its angular velocity increases infinitely, so that the circulation remains constant. It is clear that for such a model a singularity of the peripheral velocity, $v_\theta$, remains inherent. For the other components of the velocity, $v_R$ and $v_\alpha$, the requirement of boundedness is inherent, since it is in agreement with the conditions for a limiting transition.

In principle, however, the development of an "induced" singularity as a result of a limiting transition is not excluded. Therefore, while solutions of the problem with only "inherent" singularities do not exist, as occurs in the case under consideration with large Reynolds numbers, as a way out of the situation we can admit of a combination of an induced singularity and the minimally possible intensity (minimal so that any excess in it will correspond to another real model, in which this excess intensity is generated naturally as the result of a limiting transition). The requirement of a minimal singularity of the function $v_R$ permits choosing the parameter P, and finding more exactly the dependence of $P^*(k^*)$, which can be determined as follows. With $0 < k < 2.86$, $P^* = 1$, since in this range there exists a finite solution; with $2.86 < k^* < \infty$, a singular jet develops in the liquid near the axis, and the dependence of $P^*(k^*)$ should correspond to the curve on Fig. 1 from [7], separating zone B from the region in which solutions do not exist. Such a choice was proposed in [7], but only for turbulent flow with a self-adjusting virtual viscosity. The establishment of the dependence of $P^*(k^*)$ renders the problem uniquely determined. However, for this assertion to hold, the singularity of the solution with fixed values of $P^*$ and $k^*$ must be proven. Such a proof was obtained neither in [1] nor in [7]. It is set out below for the case $P^* = 1$ and those values of $k^*$ with which a solution exits.

From (5), after integration we obtain the relationship

$$\Omega' = a \exp\left(-2\int_0^x f dx\right) \tag{9}$$

where $a$ is an integration constant subject to determination from the condition $\Omega(1) = C$.

We remove this condition and shall assume that $a = \Omega'(0)$ is a given number.

As is shown in [1], $0 < a < C$. Let us consider the system of equations (4), (6), (9). We shall show that, for any given value of $a$ for which this system is solvable in the interval (0, 1), the solution is unique. We write (6) in the form

$$G(x) = x^2 - Ax - 2\int_0^x \frac{(x-t)(1-tx)}{(1-t^2)^2}\Omega^2 dt \tag{10}$$

$$\left(A = 1 - 2\int_0^1 \frac{\Omega^2 dt}{(1+t)^2}\right)$$

With such a definition of the quantity A, we have $G(1) = 0$. We discard the latter equality, and along with it also the condition $F(1) = 0$, selecting A arbitrarily. In this case, the system of equations (4), (9), (10) is equivalent to the Cauchy problem for the system (2)-(3) with the conditions

$$\Omega = F = F' = 0, \quad \Omega' = a, \quad F'' = A, \quad F''' = 1/\nu \quad \text{at} \quad x = 0$$

Such a problem is uniquely solvable. We shall show that, with a rise in the value of A, the function G(x) decreases monotonically. Let

$$A = A_0 + A_1, \quad \Omega = \Omega_0 + \Omega_1, \quad G = G_0 + G_1, \quad f = f_0 + f_1$$

where the quantities with the subscript 1 are small, $A_1 > 0$.

From (10) we find

$$G_1 = -A_1 x - 4\int_0^x \frac{(x-t)(1-tx)}{(1-t^2)^2}\Omega_0\Omega_1 dt \tag{11}$$

Differentiating this equality, it can be shown that, in some interval, $G_1 < 0$, regardless of the sign of the function $\Omega_1$. Then, in this interval, $f_1 < 0$, since $f_1$ satisfies the equation

$$f_1' = -2f_0 f_1 + k^2 G_1 / (1 - x^2)^2 \tag{12}$$
$$(f_0 \leqslant 0, \quad f_1(0) = 0)$$

Further, from (9) it follows that

$$\Omega_1' = -2\Omega_0' \int_0^x f_1 dx \qquad (\Omega_0' > 0)$$

Therefore, in the interval under consideration, $\Omega_1 > 0$. This inequality permits broadening the interval in which $G_1$ takes on negative values to the whole region of existence of a solution.

The monotonic character of the dependence of G on A permits asserting the singularity of the roots of the function G(1), and, together with this, also the uniqueness of the solution of the auxiliary boundary-value problem with a fixed value of $a$.

Further, let G again be determined by formula (6) with $P = 1$. We shall show that the function $\Omega_1(x)$ increases monotonically everywhere with a rise in the value of the constant $a$. We set

$$a = a_0 + a_1, \quad \Omega = \Omega_0 + \Omega_1, \quad G = G_0 + G_1, \quad f = f_0 + f_1, \quad a_1 > 0$$

We have

$$\Omega_1' = \left(\frac{a_1}{a_0} - 2\int_0^x f_1 dx\right)\Omega_0' \tag{13}$$

$$\left(\frac{G_1}{x}\right)' = -4\,\frac{1-x^2}{x^2}\int_0^x \frac{t\Omega_0\Omega_1 dt}{(1-t^2)^2} \tag{14}$$

It follows from (13) that, in some interval, $\Omega_1 > 0$; then, from (14) it follows that $G_1 < 0$. In this case in accordance with (12), $f_1 < 0$, and the inequality $\Omega_1 > 0$ may again be established for the whole interval of existence of a solution to the problem. As a consequence of the monotonic character of the dependence of $\Omega$ on $a$, the equality $\Omega(1) = C$ can be reached only with a unique value of $a$, which completes the proof of the theorem of uniqueness for the case $P = 1$. We note that, for $P < 1$, the proof is somewhat more complicated.

Thus, the work of Serrin [7] and its interpretation given in the present article permit resolving the paradox of [1].

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